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ON EXTREMAL QUASICONFORMAL EXTENSIONS OF CONFORMAL MAPPINGS[†]

BY
EDGAR REICH

ABSTRACT
Let $f(t,z)=z+t\omega(1/z)$ be schlicht for $|z|>1$, $\omega(z)=\sum_{n=0}^{\infty}a_nz^n$, $t>0$. The paper considers first-order estimates for the dilatation of extremal quasiconformal extensions of f as $t\to 0$.

1. Suppose

$$f(z)=z+a_0+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots,\qquad |z|>1 \tag{1.1}$$

is schlicht. Depending on the coefficients $\{a_n\}$ there may or may not [2, 4] exist a quasiconformal extension of f to the complex plane. If a quasiconformal extension f exists let

$$\kappa(z)=f_{\bar z}/f_z$$

be its complex dilatation, and let

$$k=\|\kappa\|_\infty=\operatorname*{ess\,sup}_{z\in U}|\kappa(z)|,\qquad (U=\{|z|<1\}).$$

There then exists an *extremal* extension f^* of f such that the corresponding dilatation $\kappa^*(z)$ results in the minimal possible $\|\kappa^*\|_\infty$. Although the quantity

$$k^*=\|\kappa^*\|_\infty$$

is, in principle, determined by $\{a_n\}$, relatively little concrete knowledge of the relationship exists to date. In contrast with norming the complex dilatation with the ess sup norm, the L^2 norm,

$$\|\kappa\|_2=\left[\frac{1}{\pi}\int\int_U|\kappa(z)|^2dxdy\right]^{1/2} \tag{1.2}$$

may be employed. If f is quasiconformal in the plane and agrees with (1.1) for $|z|>1$ we shall say that f is *mean-extremal* if (1.2) is minimized. Let k_2^* denote

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the infinum of values of $\|\kappa\|_2$ thus obtained. k_2^* is also determined by $\{a_n\}$, and, of course, $k_2^* \leqq k^*$.

In the literature (see especially [2]; in [6] a different normalization is used) upper bounds for various combinations of coefficients of functions of type (1.1) have been found on the assumption that a quasiconformal extension exists. Thus, for example, Kühnau [2] and Lehto [3] generalized the Bieberbach area theorem to show that if f has a quasiconformal extension with $\|\kappa\|_\infty = k$ then

$$\sum_{n=1}^{\infty} n|a_n|^2 \leqq k^2.$$

It follows, of course, that

$$k^* \geqq \sqrt{\sum_{n=1}^{\infty} n|a_n|^2}.$$

Similarly, other estimates for k^* from *below* are implied by inequalities for other combinations of coefficients found in [2].

In what follows we will restrict ourselves to the case when (1.1) is an *infinitesimal deformation of the identity*; that is, f is of the form

$$f(t,z) = z + t\omega\left(\frac{1}{z}\right), \qquad |z|>1. \tag{1.3}$$

Here $t>0$ is a parameter which approaches 0, and

$$\omega(z) = \sum_{n=0}^{\infty} a_n z^n$$

converges in U. In fact, we shall assume that

$$M = \sup_{z\in U} |\omega'(z)| < \infty, \qquad 0 \leqq t < \frac{1}{M}, \tag{1.4}$$

insuring[†] automatically that $f(t,z)$ is continuous and 1–1 for $|z| \geqq 1$, since for any function g, analytic in U,

$$\frac{g(z_2)-g(z_1)}{z_2-z_1} = \int_0^1 g'(tz_2+(1-t)z_1)dt, \qquad z_1, z_2 \in U.$$

Let $k^*(t)$, $k_2^*(t)$ denote the respective values of k^* and k_2^* corresponding to the parameter t and the given sequence $\{a_n\}$. We shall determine the values of $k^*(t)$, and $k_2^*(t)$ up to a term of order $o(t)$, $t \to 0$. In the case of $k^*(t)$ the "determination" (Theorem 1) is in terms of an equivalent (but as yet largely

[†] The author wishes to thank the referee for pointing out the advantage of formulating condition (1.4) in the present form.

unresolved) coefficient problem for a class of analytic functions. In the case of $k_2^*(t)$ the result (Theorem 2) is especially transparent.

2. We recall a general method [5] for determining $k^*(t)$ asymptotically as $t \to 0$ for mappings of U with specified quasiconformally realizable boundary values. Let $g(t, z)$, $t > 0$, be a one-parameter family of quasiconformal mappings of U onto U having complex dilatations

$$\kappa(t, z) = g_{\bar{z}}/g_z$$

of the form

$$\kappa(t, z) = t\mu(z) + o(t), \; t \to 0, \; z \in U, \tag{2.1}$$

where $\mu(z)$ is an essentially bounded measurable function and $o(t)$ is uniform with respect to z. Here $k^*(t)$ denotes the minimal dilatation achievable with a quasiconformal mapping having the same boundary values on ∂U as $g(t, z)$. If the image domain $g(t, U)$ is some Jordan domain other than U this does not affect any relation between $k^*(t)$ and $\kappa(t, z)$ since both are invariant under composition with conformal mapping.

Let $\mathfrak{B} = \{\varphi(z)\colon \varphi(z)$ is holomorphic in U, $\|\varphi\| = \iint_U |\varphi(z)| dxdy < \infty\}$, and let $H[\mu]$ denote the nonnegative number

$$H[\mu] = \sup_{\varphi \in \mathfrak{B}} \frac{\left| \iint_U \mu(z)\varphi(z)\, dxdy \right|}{\|\varphi\|}. \tag{2.2}$$

Then [5, theorem 13]

$$k^*(t) = tH[\mu] + o(t), \qquad t \to 0. \tag{2.3}$$

To apply the above result to the problem of determining the asymptotic behavior of the $k^*(t)$ of §1 one evidently needs to have *some* family of quasiconformal mappings $g(t, z)$ of U whose boundary values on ∂U agree with those of $f(t, z)$. Under our hypothesis (1.4) one such mapping is

$$g(t, z) = z + t\omega(\bar{z}), \qquad z \in U. \tag{2.4}$$

In fact, if $g(t, z)$ is defined by (2.4) then the complex dilatation is immediately seen to have the desired form (2.1), with

$$\mu(z) = \omega'(\bar{z}). \tag{2.5}$$

3. In order to proceed we formulate the extremal problem for linear combinations of coefficients for functions

$$(3.1)\qquad \varphi(z)=\sum_{n=0}^{\infty} c_n z^n, \qquad |z|<1$$

of class $\mathfrak{B}$.

If $\lambda_0, \lambda_1, \cdots, \lambda_p$ are any complex numbers, let

$$F(\lambda_0, \lambda_1, \cdots, \lambda_p)=\sup_{\varphi\in\mathfrak{B}} \frac{\left|\sum_{n=0}^{p} \lambda_n c_n\right|}{\|\varphi\|}.$$

Since

$$\{\varphi\in\mathfrak{B}:\|\varphi\|\leqq 1\}$$

is normal and compact an extremal φ for F exists. For the case

$$\lambda_i=\delta_{im}=\begin{cases}1, & i=m\\ 0, & i\neq m\end{cases}$$

$$(3.2)\qquad F(\{\delta_{im}\})=\sup_{\varphi\in\mathfrak{B}}\frac{|c_m|}{\|\varphi\|}=\frac{m+2}{2\pi}, \qquad m=0,1,2,\cdots,$$

and $\varphi(z)=z^m$ is extremal. This is seen by integrating Cauchy's formula. Moreover, if $\{\lambda_i\}$ is an infinite sequence, then (3.2) implies

$$(3.3)\qquad F(\lambda_0, \lambda_1, \cdots,)\leqq \sum_{n=0}^{\infty}\frac{n+2}{2\pi}|\lambda_n|.$$

Our conclusion is now the following.

THEOREM 1. *Let $f(t,z)$ be the conformal mapping* (1.3) *for which* (1.4) *holds. Let $k^*(t)$ denote the dilatation of the corresponding extremal quasiconformal extension. Then*

$$(3.4)\qquad \lim_{t\to 0}\frac{k^*(t)}{t}=\pi F(a_1, a_2, \cdots,).$$

In particular, as $t\to 0$,

$$(3.5)\qquad k^*(t)\leqq \frac{t}{2}\sum_{n=1}^{\infty}(n+1)|a_n|+o(t).$$

PROOF. Using (2.5) and (3.1) one obtains

$$\iint_U \mu(z)\varphi(z)dxdy=\sum_{n=1}^{\infty} na_n\iint_U \bar{z}^{n-1}\varphi(z)dxdy=\pi\sum_{n=1}^{\infty}a_n c_{n-1}.$$

Thus (2.2) becomes

$$H[\mu] = \pi F(a_1, a_2, \cdots,),$$

and the assertions (3.4) and (3.5) follow from (2.3) and (3.3) respectively.

In view of (3.2), in the special case

$$f(t, z) = z + ta_m/z^m, \tag{3.6}$$

we obtain the exact asymptotic formula,

$$k^*(t) = t\frac{m+1}{2} |a_m| + o(t), \qquad t \to 0. \tag{3.7}$$

Thus the coefficient $t/2$ in (3.5) is best possible. (Note that (3.7) is evidently related to, although not obviously equivalent to, a special case of [2, equation (67)].) It is worthwhile comparing (3.5) with the estimate

$$k^*(t) \leqq tM \leqq t\sum_1^{\infty} n|a_n|$$

which follows trivially from (2.4).

An explicit determination of $F(\lambda_0, \lambda_1, \cdots,)$ together with a determination of the corresponding extremal functions of $\mathfrak{B}$ would evidently be of interest.

4. We next consider the problem of characterizing mean-extremal mappings with complex dilatations of the form (2.1), $t \to 0$. Let $\mathfrak{N}$ be the collection of essentially bounded complex valued measurable functions $\nu(z)$, $z \in U$, such that

$$\int\int_U \nu(z)\varphi(z)dxdy = 0 \tag{4.1}$$

for all $\varphi \in \mathfrak{B}$. A family of mappings of the plane which is normalized as for (1.3) and which has a complex dilatation

$$t\nu(z) + o(t), \qquad \nu \in \mathfrak{N},$$

keeps ∂U pointwise fixed up to order $o(t)$, $t \to 0$, [1]. If $g_j(t, z)$, $j = 1, 2$, are mappings of the plane with dilatations respectively of the form

$$\kappa_j(t, z) = t\mu_j(z) + o(t), \qquad j = 1, 2,$$

then $g_2 \circ g_1$ has dilatation

$$t[\mu_1(z) + \mu_2(z)] + o(t).$$

Therefore a mapping with dilatation of the form (2.1) will be mean extremal (up to composition with a correction mapping which only affects the $o(t)$ term) if and only if

$$\min_{\nu \in \mathfrak{N}} \|\mu + \nu\|_2 = \|\mu\|_2. \tag{4.2}$$

If μ satisfies (4.2) then

$$k_2^*(t) = \|\mu\|_2 t + o(t). \tag{4.3}$$

LEMMA. *Let $\mu(z)$ be an essentially bounded complex valued measurable function, $z \in U$. A necessary and sufficient condition for* (4.2) *to hold is that $\overline{\mu(z)}$ is analytic in U.*

PROOF. Suppose (4.2) holds. Let $\mu(z)$ have Fourier series

$$\mu(re^{i\theta}) \sim \sum_{n=1}^{\infty} A_n(r)e^{in\theta} + \sum_{n=0}^{\infty} B_n(r)e^{-in\theta}.$$

Consider

$$\nu(re^{i\theta}) = A_l(r)e^{il\theta} + \beta_m(r)e^{-im\theta},$$

where $l \geqq 1$, $m \geqq 0$, and where $\beta_m(r)$ is a bounded function,

$$\int_0^1 r^{m+1}\beta_m(r)dr = 0. \tag{4.4}$$

Condition (4.4) guarantees that $\nu \in \mathfrak{N}$ since it is equivalent to

$$\iint_U z^n \nu(z)dxdy = 0, \qquad n = 0, 1, 2, \cdots.$$

By (4.2),

$$0 = \iint_U \overline{\mu(z)}\nu(z)dxdy = 2\pi\int_0^1 [|A_l(r)|^2 + \overline{B_m(r)}\beta_m(r)]rdr. \tag{4.5}$$

Since (4.5) holds for any bounded $\beta_m(r)$ satisfying (4.4), we conclude that $A_l(r) = 0$, $l = 1, 2, \cdots$, and that

$$B_m(r) = C_m r^m. \tag{4.6}$$

Conversely, every element of $\mathfrak{N}$ has Fourier series

$$\nu(re^{i\theta}) \sim \sum_{n=1}^{\infty} \alpha_n(r)e^{in\theta} + \sum_{n=0}^{\infty} \beta_n(r)e^{-in\theta}$$

such that $\alpha_n(r)$, $\beta_n(r)$ are bounded functions subject to (4.4). If $\overline{\mu(z)}$ is bounded and analytic in U one sees immediately that

$$\iint_U \overline{\mu(z)}\nu(z)dxdy = 0.$$

THEOREM 2. *Let $f(t, z)$ be the conformal mapping* (1.3) *for which* (1.4) *holds. Let $k_2^*(t)$ be the L^2 mean of the complex dilatation of a mean-extremal extension of f. Then*

$$\lim_{t\to 0} \frac{k_2^*(t)}{t} = \| \omega'(\bar{z}) \|_2 = \sqrt{\sum_{n=1}^{\infty} n|a_n|^2}.$$

PROOF. In view of the lemma, for an extension of (1.3) to be mean extremal it is necessary that the complex dilatation have the form

$$\kappa(t, z) = \overline{th(z)} + o(t), \qquad z \in U$$

where $h(z)$ is a bounded analytic function, $z \in U$. Conversely, if an extension of $f(t, z)$ has a complex dilatation of the above form, then, by the lemma, and (4.3),

$$\lim_{t\to 0} \frac{k_2^*(t)}{t} = \| h \|_2.$$

In the present case we may, as per (2.4), (2.5), take

$$h(z) = \overline{\omega'(\bar{z})}.$$

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SCHOOL OF MATHEMATICS
UNIVERSITY OF MINNESOTA
MINNEAPOLIS, MINNESOTA 55455 USA